

ON REAL ONE-SIDED IDEALS IN A FREE ALGEBRA

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ABSTRACT. In classical and real algebraic geometry there are several notions of the radical of an ideal I . There is the ordinary radical \sqrt{I} and the real radical $\sqrt[re]{I}$. This paper and a previous one focus on extensions of these and their benefits to the free $*$ -algebra $\mathbb{R}\langle x, x^* \rangle$ of noncommutative real polynomials in $x = (x_1, \dots, x_g)$ and $x^* = (x_1^*, \dots, x_g^*)$.

We work with a natural notion of the (noncommutative) zero set $V(I)$ of a left ideal I in $\mathbb{R}\langle x, x^* \rangle$. The radical \sqrt{I} of I is the set of all $p \in \mathbb{R}\langle x, x^* \rangle$ which vanish on $V(I)$. In this paper our quest is to find classes of left ideals I which coincide with their radical \sqrt{I} ; such ideals are called radical. We completely succeed for monomial ideals and homogeneous principal ideals. For several other classes of ideals we have partial success. Also we give an algorithm (running under [NCAgebra](#)) which checks if a left ideal is radical or is not, and illustrate how one uses our implementation of it.

An earlier paper [[CHMN+](#)] gives an appropriate notion of $\sqrt[re]{I}$ and proves $\sqrt{I} = \sqrt[re]{I}$, a free $*$ -Nullstellensatz. Our methods here make heavy use of this result.

1. INTRODUCTION

The introduction begins with definitions and a little motivation for them. Then it sketches the main results of this paper together with links to where they are found in this paper.

1.1. Zero Sets in Free Algebras. Let $\langle x, x^* \rangle$ be the monoid freely generated by $x = (x_1, \dots, x_g)$ and $x^* = (x_1^*, \dots, x_g^*)$, i.e., $\langle x, x^* \rangle$ consists of **words** in the $2g$ noncommuting letters $x_1, \dots, x_g, x_1^*, \dots, x_g^*$ (including the empty word \emptyset which plays the role of the identity 1). Let $\mathbb{R}\langle x, x^* \rangle$ denote the \mathbb{R} -algebra freely generated by x, x^* , i.e., the elements of $\mathbb{R}\langle x, x^* \rangle$

Date: August 24, 2012.

2010 *Mathematics Subject Classification.* Primary 14P10, 08B20; Secondary 90C22, 16W10, 13J30.

Key words and phrases. real algebraic geometry, Nullstellensatz, real ideal, noncommutative polynomial.

¹Research supported by the grant P1-0222 from the Slovenian Research Agency, ARRS.

²Research supported by NSF grants DMS-0700758, DMS-0757212, and the Ford Motor Co.

³Supported by the Faculty Research Development Fund (FRDF) of The University of Auckland (project no. 3701119). Partially supported by the Slovenian Research Agency grant P1-0222.

⁴Research supported by NSF grants DMS-0758306 and DMS-1101137.

⁵Research supported by NSF grants DMS-0700758, DMS-0757212.

are **polynomials** in the noncommuting variables x, x^* with coefficients in \mathbb{R} . Equivalently, $\mathbb{R}\langle x, x^* \rangle$ is the **free *-algebra** on x . The length of the longest word in a noncommutative polynomial $f \in \mathbb{R}\langle x, x^* \rangle$ is the **degree** of f and is denoted by $\deg(f)$. The set of all words of degree at most k is $\langle x, x^* \rangle_k$, and $\mathbb{R}\langle x, x^* \rangle_k$ is the vector space of all noncommutative polynomials of degree at most k .

There are several possible notions of zero of a noncommutative polynomial and we shall give one which has proven productive in a variety of situations, e.g. [HMP07, HM+]. Given a g -tuple $X = (X_1, \dots, X_g)$ of same size square matrices over \mathbb{R} , write $p(X)$ for the natural evaluation of p at X . For $S \subseteq \mathbb{R}\langle x, x^* \rangle$ we introduce

$$V(S)^{(n)} = \{(X, v) \in (\mathbb{R}^{n \times n})^g \times \mathbb{R}^n \mid p(X)v = 0 \text{ for every } p \in S\},$$

and define the **zero set** of S to be

$$V(S) = \bigcup_{n \in \mathbb{N}} V(S)^{(n)} = \{(X, v) \mid p(X)v = 0 \text{ for every } p \in S\}.$$

To each subset T of $\bigcup_{n \in \mathbb{N}} ((\mathbb{R}^{n \times n})^g \times \mathbb{R}^n)$ we associate the **left ideal**

$$\mathcal{I}(T) = \{p \in \mathbb{R}\langle x, x^* \rangle \mid p(X)v = 0 \text{ for every } (X, v) \in T\}.$$

For a left ideal I of $\mathbb{R}\langle x, x^* \rangle$, we call

$$\sqrt{I} := \mathcal{I}(V(I))$$

the **radical** of I . Evidently \sqrt{I} is a left ideal. We say that I is **has the left Nullstellensatz property** if $\sqrt{I} = I$. Now we describe a class of ideals which has this property.

A polynomial p is **analytic** if it has no transpose variables, that is, no x_i^* . For example, $p(x) = 1 + x_1x_2 + x_1^3 + x_2^5$ is analytic while $p(x) = 1 + x_1^*x_2 + (x_1^*)^3 + x_2^5$ is not analytic. There is a strong Nullstellensatz for left ideals generated by analytic polynomials in [HMP07]. This strengthens an earlier result proved by Bergman [HM04]; see also [BK11] for a survey on noncommutative Nullstellensätze.

Theorem 1.1. *Let p_1, \dots, p_m be analytic polynomials, and let I be the left ideal generated by those p_i . Then $\sqrt{I} = I$, i.e.,*

$$(\forall j \ p_j(X)v = 0) \implies q(X)v = 0 \quad \text{iff} \quad q = f_1p_1 + \dots + f_mp_m.$$

For this theorem we observe two things. No powers q^k are needed, contrary to the case in the classical commutative Hilbert Nullstellensatz or the real Nullstellensatz. This absence of powers seems to be the pattern for free algebra situations. Another point is that using pairs (X, v) in our definition of zeroes gives a clean result. While defining zero of a noncommutative polynomial p by $p(X) = 0$ does not give a strong result (this notion of zero is too coarse). $\det p(X) = 0$ is better, but does not seem to lead as far. We also mention

that our notion of zero plays a key role in [HM+] and is used to prove a result unrelated to Nullstellensätze.

What about ideals generated by p_j which are not analytic? To shed light on the basic question of which ideals have the left Nullstellensatz property, we seek an algebraic description of the radical \sqrt{I} similar to the notion of real radical in the classical real algebraic geometry, cf. [BCR98, Chapter 4], [Mar08, Chapter 2], [PD01, Chapter 4] or [Sce09]. For this we introduce real ideals.

A left ideal I of $\mathbb{R}\langle x, x^* \rangle$ is said to be **real** if for every $a_1, \dots, a_r \in \mathbb{R}\langle x, x^* \rangle$ such that

$$\sum_{i=1}^r a_i^* a_i \in I + I^*,$$

we have that $a_1, \dots, a_r \in I$. Here I^* is the right ideal $I^* = \{a^* \mid a \in I\}$. An intersection of a family of real ideals is a real ideal. For a left ideal J of $\mathbb{R}\langle x, x^* \rangle$ we call the ideal

$$\sqrt[re]{J} = \bigcap_{\substack{I \supseteq J \\ I \text{ real}}} I = \text{the smallest real ideal containing } J$$

the real radical of J . The main result of [CHMN+] states:

Theorem 1.2 ([CHMN+, Theorem 1.6]). *A finitely generated left ideal in $\mathbb{R}\langle x, x^* \rangle$ satisfies the left Nullstellensatz property if and only if it is real.*

Also there is a quantitative version of this theorem that gives us bounds (which we shall need) on the degrees of the polynomials involved.

Theorem 1.3 ([CHMN+, Theorem 2.6]). *Let I be a left ideal in $\mathbb{R}\langle x, x^* \rangle$ generated by polynomials of degree bounded by d . Then I is real if and only if whenever q_1, \dots, q_k are polynomials with $\deg(q_j) < d$ for each j , and $\sum_{i=1}^{\ell} q_i^* q_i \in I + I^*$, then $q_j \in I$ for each j .*

These results give a clean equivalence but do not tell us whether or not a particular ideal has the left Nullstellensatz property. This paper focuses on examples of ideals I for which we can determine if $I = \sqrt{I}$.

1.2. Main Results. In this paper we consider two classes of ideals, monomial ideals and homogeneous principal ideals and give simple tests which determine exactly which of these do and which do not have the left Nullstellensatz property. Also we describe in Section 4 a computer algorithm to determine if a given I is a real ideal, hence if I has the left Nullstellensatz property. This is a more practical refinement of a special case of an algorithm given in [CHMN+]. We have implemented it and illustrate its use. As a tool for future use in actually computing the real radical of an ideal we describe an algorithm for generating a Gröbner basis for a left ideal.

Our main result for left monomial ideals is easily stated. A **monomial ideal** is a left ideal generated by monomials. A word $w \in \langle x, x^* \rangle$ is **left unshrinkable** [Lan97, Tap99] if it cannot be written as $w = uu^*d$ for some $d, u \in \langle x, x^* \rangle$ with $u \neq 1$. We show:

Theorem 1.4. *A finitely generated left monomial ideal is real if and only if it is generated by left unshrinkable words if and only if it has the left Nullstellensatz property.*

The first iff is proved in Section 2 and does not require finite generation. Combine this with Theorem 1.2 to get the second iff for finitely generated ideals. Absent finitely many generators we need something more to prove a version of the last iff, see Sections 5 and 6. Section 6 concerns something of a dual to free real ideals. Such duals have powerful uses, for example they are used heavily in proving Theorem 1.2, and here we use them to yield a version of Theorem 1.2 when there are infinitely many generators; it is Corollary 6.2.

Section 3 focuses on principal ideals and for homogeneous ones we get a definitive result. Our result for homogeneous principal ideals, Theorem 3.11, requires several definitions to state. The core part of these results is Proposition 3.19 which states that the left ideal $I \subseteq \mathbb{R}\langle x, x^* \rangle$ generated by a polynomial p with highest degree homogeneous part p' is real only if none of the $2k$ polynomials

$$\pm(p_1 + p_1^*), \pm(p_1p_2 + p_2^*p_1^*), \dots, \pm(p_1 \cdots p_k + p_k^* \cdots p_1^*)$$

is a sum of squares. Here, $p' = p_1p_2 \cdots p_k$ is a factorization of p' into irreducible factors. By Corollary 3.10, for the left ideal $I \subseteq \mathbb{R}\langle x, x^* \rangle$ generated by a homogeneous polynomial p this condition is necessary and sufficient for I to be real. Factoring algorithms like Algorithm 3.6 and Remark 3.12 make the condition in this theorem checkable; independent of our general algorithm in Section 4.

For a nonhomogeneous principal ideal determining if it is radical seems problematic. Even for an ideal I with a one variable degree two generator p the p making the ideal have the left Nullstellensatz property do not seem to present an obvious pattern, see Subsection 4.4. In Subsection 3.3 we present further classes of (principal) left ideals which are real.

2. MONOMIAL IDEALS

In this section we prove Theorem 1.4.

Lemma 2.1. *For a left ideal $I \subseteq \mathbb{R}\langle x, x^* \rangle$ the following are equivalent:*

- (i) *I is a left monomial ideal;*
- (ii) *A polynomial $f \in \mathbb{R}\langle x, x^* \rangle$ is in I iff all of the monomials appearing in f are in I .*
- (iii) *A polynomial $f \in \mathbb{R}\langle x, x^* \rangle$ is in $I + I^*$ iff all of the monomials appearing in f are in $I + I^*$.*

(iv) A polynomial $f \in \mathbb{R}\langle x, x^* \rangle$ is in $I + I^*$ iff all of the monomials appearing in f are in I or I^* .

Proof. (i) \Rightarrow (ii) is obvious. If (ii) holds, then I is generated by the set of all monomials in I so (i) holds. The equivalence of (i) and (iii) follows similarly. It is clear that (iii) and (iv) are equivalent. \blacksquare

Proof of Theorem 1.4. Suppose I is generated by left unshrinkable words $\{w_i \mid w_i \in \langle x, x^* \rangle, i \in G\}$ for some index set G . Consider a sum of hermitian squares

$$s = \sum_j g_j^* g_j$$

and decompose each g_j as

$$g_j = \sum_{w \in \langle x, x^* \rangle} a_{w,j} w = \sum_{\substack{w \in \langle x, x^* \rangle \\ w \in I}} a_{w,j} w + \sum_{\substack{u \in \langle x, x^* \rangle \\ u \notin I}} a_{u,j} u = p_j + q_j$$

for some $a_{v,k} \in \mathbb{R}$. Here $p_j = \sum_{w \in I} a_{w,j} w \in I$ and $q_j = \sum_{u \notin I} a_{u,j} u \notin I$. Then

$$s = \sum_j g_j^* g_j = \sum_j (p_j^* + q_j^*)(p_j + q_j) = \sum_j (p_j^* p_j + p_j^* q_j + q_j^* p_j + q_j^* q_j)$$

being in $I + I^*$ is equivalent to

$$(1) \quad \sum_j q_j^* q_j \in I + I^*.$$

Assume I is not real, and let s be a sum of squares of the form (1) of minimal degree with not all q_j in I . Since the highest degree terms in a sum of squares cannot cancel, some of the highest degree terms in s are multiples of words of the form $w^* w$ for words $w \notin I$. Lemma 2.1 implies that $w^* w \in I + I^*$ for each such w . Since I is a left monomial ideal, it follows that $w^* w$ is of the form $v_1 v_2$, where v_2 is one of the unshrinkable words which generate I .

If $\deg(w) \geq \deg(v_2)$, then this implies that $w = v' v_2$, where v' is the right-hand piece of v_1 , which is a contradiction since $w \notin I$.

If $\deg(w) < \deg(v_2)$, then $v_2 = w' w$, where w' is the left-hand piece of v_2 , and $\deg(w') > 0$. Thus $v_1 v_2 = v_1 w' w = w^* w$, which implies that $w^* = v_1 w'$, which implies that $w = (w')^* v_1^*$. Therefore $v_2 = w'(w')^* v_1^*$, which is a contradiction since v_2 is unshrinkable.

Conversely, suppose I is a real left monomial ideal. Let \mathcal{B} be a set of words which generate I . Without loss of generality, we may assume that for each $w \in \mathcal{B}$, there exist no $u, v \in \mathbb{R}\langle x, x^* \rangle$, with $v \in I$, such that $w = uv$. Assume $w \in \mathcal{B}$ is shrinkable. Let $w = u^* uv$, with u some nonconstant word. Thus $v^* u^* uv \in I + I^*$, which implies that $uv \in I$ by I being real. This however contradicts the minimality of \mathcal{B} . \blacksquare

In Section 6 we return to the theme of monomial ideals. There we construct a separating strictly positive linear functional for each (not necessarily finitely generated) real monomial left ideal, and use it to produce a $*$ -representation. Our results on monomial ideals can be viewed in the context of one-sided analogs of the Lance-Tapper [Lan97, Tap99] conjecture (see also [Pop10]).

3. PRINCIPAL LEFT IDEALS

We now turn our attention to principal (i.e., singly generated) left ideals. Our focus here is on homogeneous generators. The main results in this section are Theorem 3.11 that precisely describes when a principal homogeneous left ideal is real, and the accompanying Algorithm 3.6, producing a practical test for determining whether a given homogeneous polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ generates a real left ideal.

3.1. Factorization of Homogeneous Polynomials. Checking whether a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ factors as a product $p = p_1 p_2$ of polynomials of smaller degree amount to solving a large system of linear equations. For homogeneous noncommutative polynomials factorization can be done more efficiently, and this is what we describe in this subsection.

Lemma 3.1. *Let $V(x) = (v_1, \dots, v_r)^* \in \mathbb{R}\langle x, x^* \rangle^r$ be a vector of linearly independent homogeneous degree d_r polynomials, and let $W(x) = (w_1, \dots, w_s)^* \in \mathbb{R}\langle x, x^* \rangle^s$ be a vector of linearly independent homogeneous degree d_s polynomials. For $A \in \mathbb{R}^{r \times s}$, we have*

$$V(x)^* A W(x) = 0 \quad \Longleftrightarrow \quad A = 0.$$

Proof. First consider the case where $V = M_{d_r}$ is a vector whose entries are all the words of degree d_r , and where $W = M_{d_s}$ is a vector whose entries are all the words of degree d_s . Let $A \in \mathbb{R}^{r \times s}$ satisfy $M_r(x)^* A M_s(x) = 0$. Each word w of degree $d_r + d_s$ can be expressed uniquely as a product

$$w = w_r w_s,$$

where w_r is the word consisting of the first d_r letters of w and w_s is the word consisting of the last d_s letters of w . Let $A_{w_r^*, w_s}$ be the entry of A corresponding to w_r^* on the left and w_s on the right. Then,

$$M_r(x)^* A M_s(x) = \sum_{\deg(w_r)=d_r} \sum_{\deg(w_s)=d_s} A_{w_r^*, w_s} w_r w_s = 0.$$

This implies that each $A_{w_r^*, w_s} = 0$, or in other words, that $A = 0$.

Next consider the case where the entries of V span the set of all homogeneous degree d_r polynomials, and where the entries of W span the set of all homogeneous degree d_s

polynomials. The elements of V may be expressed uniquely as a linear combination of the elements of M_r . Therefore there exists an invertible matrix R such that

$$RV(x) = M_r(x).$$

Similarly, there exists an invertible matrix S such that

$$SW(x) = M_s(x).$$

Therefore

$$V(x)^*AW(x) = M_r^*(R^{-1})^*AS^{-1}M_s = 0$$

which implies that $(R^{-1})^*AS^{-1} = 0$. Since R and S are invertible, it follows that $A = 0$.

Finally, consider the general case. Let V' be a vector whose entries, together with the entries of V , form a basis for the set of homogeneous degree d_r polynomials, and let W' be a vector whose entries, together with the entries of W , form a basis for the set of homogeneous degree d_s polynomials. Suppose $V(x)^TAW = 0$. Then

$$V^TAW = \begin{bmatrix} V \\ V' \end{bmatrix}^T \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W \\ W' \end{bmatrix} = 0,$$

which implies that $A = 0$ by previously established case. ■

Remark 3.2. Let $M_{d_1} = (w_1^{(1)}, \dots, w_k^{(1)})$ be a vector whose entries are all words of length d_1 and $M_{d_2} = (w_1^{(2)}, \dots, w_\ell^{(2)})$ be a vector whose entries are all words of length d_2 . Let p be homogeneous of degree $d_1 + d_2$. Then the unique matrix A such that

$$p = M_{d_1}^*AM_{d_2}$$

is the matrix whose (i, j) -entry is the coefficient of $(w_i^{(1)})^*w_j^{(2)}$ in p . We call it the (d_1, d_2) -**Gram matrix** for p . (Observe that the uniqueness of the Gram matrix fails for nonhomogeneous polynomials.)

Lemma 3.3 (cf. [Har+, KP10]). Let $p \in \mathbb{R}\langle x, x^* \rangle$ be a homogeneous degree $2d$ polynomial. Then p is a sum of squares if and only if its (d, d) -Gram matrix is positive semidefinite.

Proof. If $p = M_d^*AM_d$, and $A \succeq 0$, then

$$p = \left(\sqrt{A}M_d\right)^* \left(\sqrt{A}M_d\right),$$

which implies that p is a sum of squares.

Conversely, suppose $p = p_1^*p_1 + \dots + p_k^*p_k$. Let p'_i be the terms of p_i that are of degree $\max_j\{\deg(p_j)\}$. Then

$$(2) \quad p = \sum_{i=1}^k \left((p'_i)^*(p'_i) + (p_i - p'_i)^*(p'_i) + (p'_i)^*(p_i - p'_i) + (p_i - p'_i)^*(p_i - p'_i) \right).$$

The first term of (2) yields a sum of squares of degree $2 \max_i \{\deg(p_i)\}$ and the other terms have degree $< 2 \max_i \{\deg(p_i)\}$. Since p is homogeneous, and since a sum of nonzero squares is nonzero, each of the p_i is homogeneous of degree d . For each i , let $p_i = \alpha_i^* M_d$ for some scalar vector α_i of suitable dimension. Then

$$p = M_d^* \left(\sum_{i=1}^k \alpha_i \alpha_i^* \right) M_d,$$

and so $\sum_{i=1}^k \alpha_i \alpha_i^* \succeq 0$ is the (d, d) -Gram matrix for p . ■

We call a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ **irreducible** if it cannot be written as a product of two polynomials of smaller degree, i.e., if $p = qr$ with $q, r \in \mathbb{R}\langle x, x^* \rangle$, then q is constant or r is constant.

Proposition 3.4 (cf. [Co63, Ca+]). *Let $p \in \mathbb{R}\langle x, x^* \rangle$ be a homogeneous noncommutative polynomial. Then p can be factored as*

$$p = p_1 \cdots p_k$$

for some irreducible homogeneous polynomials p_1, \dots, p_k . Further, such a representation is unique in that if $p = q_1 \cdots q_\ell$ is another decomposition, where each q_j is irreducible, then $k = \ell$ and there exist nonzero scalars $\lambda_1, \dots, \lambda_k$ such that $q_i = \lambda_i p_i$ for $i = 1, \dots, k$.

We refer the reader to [Co06] for a detailed study of factorization in free algebras, and to [GGRW05] for another take on noncommutative factorization.

It is straightforward to compute a factorization of a homogeneous noncommutative polynomial. There is an algorithm described in [Ca+]; see also [KaS93]. An alternate way of looking at the algorithm is presented in Algorithm 3.6 below.

Lemma 3.5. *A homogeneous polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ can be factored as $p = p_1 p_2$, for $\deg(p_1) = d_1$ and $\deg(p_2) = d_2$ if and only if the (d_1, d_2) -Gram matrix A for p has rank 1. Indeed, a decomposition $p = p_1 p_2$ is then given by decomposing $A = \alpha_1 \alpha_2^*$ and setting $p_1 = M_{d_1}^* \alpha_1$ and $p_2 = \alpha_2^* M_{d_2}$.*

Proof. Straightforward. ■

Algorithm 3.6. *Suppose $p \in \mathbb{R}\langle x, x^* \rangle$ is a homogeneous degree d polynomial. If its (d_1, d_2) -Gram matrix is of rank > 1 for all $d_1, d_2 \in \mathbb{N}$ with $d_1 + d_2 = d$, then p is irreducible. Otherwise choose the smallest possible d_1 producing a factorization $p = p_1 \tilde{p}_1$ as in Lemma 3.5. Then p_1 is irreducible. Repeating the procedure on \tilde{p}_1 yields a factorization $\tilde{p}_1 = p_2 \cdots p_k$, where each p_j is irreducible. Then $p = p_1 p_2 \cdots p_k$.*

3.2. Real Principal Left Ideals. There is a clean test for determining whether or not a principal left ideal is real which applies to all polynomials, not just homogeneous ones.

Proposition 3.7. *Let I be the left ideal generated by a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$. Then I is real if and only if there exists no nonzero sum of squares equal to $qp + p^*q^*$ for a polynomial $q \in \mathbb{R}\langle x, x^* \rangle$ with $\deg(q) < \deg(p)$.*

Proof. The left ideal I is equal to

$$I = \{qp \mid q \in \mathbb{R}\langle x, x^* \rangle\}.$$

A polynomial qp has $\deg(qp) = \deg(q) + \deg(p)$, which implies that I contains no nonzero polynomials of degree less than $\deg(p)$.

By Theorem 1.3, I is real if and only if there exists no sum of squares of the form

$$(3) \quad q_1^*q_1 + \cdots + q_k^*q_k \in I + I^*,$$

with $\deg(q_j) < \deg(p)$ for each j , and $q_j \notin I$. Since I is principal, as shown above there are no nonzero $q_j \in I$ with $\deg(q_j) < \deg(p)$. Further, a sum of squares of the form (3) has degree equal to $2 \max\{\deg(q_j)\}$. Therefore I is real if and only if there exists no nonzero sum of squares in $I + I^*$ with degree less than $2\deg(p)$.

[CHMN+, Proposition 2.17(i)] implies that $(I + I^*)_{2d-1} = I_{2d-1} + I_{2d-1}^*$. The set I_{2d-1} is equal to

$$I_{2d-1} = \{qp \mid \deg(q) < \deg(p)\}$$

since $\deg(qp) = \deg(q) + \deg(p)$. Therefore an element of $(I + I^*)_{2d-1}$ is of the form $q_1p + p^*q_2^*$, with $\deg(q_1), \deg(q_2) < \deg(p)$. Further, if $q_1p + p^*q_2^*$ is symmetric, then

$$q_1p + p^*q_2^* = \frac{1}{2}(q_1p + p^*q_2) + \frac{1}{2}(q_1p + p^*q_2)^* = \left(\frac{1}{2}(q_1 + q_2)\right)p + p^*\left(\frac{1}{2}(q_1 + q_2)\right)^*.$$

Therefore the set of symmetric elements of $(I + I^*)_{2d-1}$ is $\{qp + p^*q^* \mid \deg(q) < \deg(p)\}$. Hence I is real if and only if there exists no nonzero sum of squares of the form $qp + p^*q^*$ with $\deg(q) < \deg(p)$. ■

Example 3.8. *Suppose $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is the left ideal generated by a homogeneous polynomial p , and p has no terms containing $x_i x_i^*$ or $x_i^* x_i$ for any i . Then I is real.*

Proof. By Proposition 3.7, it suffices to assume that there exists a polynomial q such that $qp + p^*q^*$ is a nonzero sum of squares, with $\deg(q) < \deg(p)$. Then $\deg(qp + p^*q^*) \leq \deg(q) + \deg(p) < 2\deg(p)$. A nonzero sum of squares s necessarily contains some terms of the form w^*w , where $2\deg(w) = \deg(s)$. Such a term contains a $x_i x_i^*$ or a $x_i^* x_i$ at the $\deg(s)/2$ through $\deg(s)/2 + 1$ position. However, no term of $qp + p^*q^*$ contains such a hermitian square since p contains no such terms and $2\deg(p) > \deg(qp + p^*q^*)$. ■

Lemma 3.9. *Let p, q be homogeneous polynomials with $\deg(q) \leq \deg(p)$. Then $qp + p^*q^*$ is a sum of squares if and only if $p = rq^*$ for some polynomial r such that $r + r^*$ is a sum of squares.*

Proof. Suppose $\deg(qp + p^*q^*) = 2d$ so that $d \leq \deg(p)$. Let $w_1q^*, \dots, w_kq^*, r_1, \dots, r_\ell$ be a basis for homogeneous polynomials of degree d so that w_1, \dots, w_k are all words of length $d - \deg(q)$. Then there exists a unique decomposition of qp as

$$qp = \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}^* \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}$$

for some block matrices A and B . Further,

$$qp + p^*q^* = \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}^* \begin{bmatrix} A + A^* & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}.$$

Since $qp + p^*q^*$ is a sum of squares, and by uniqueness of the Gram matrix representation, we have $B = 0$. Therefore

$$qp = \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix} = q \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}^* A \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix} q^*$$

Hence

$$r = \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}^* A \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}$$

gives $p = rq^*$ and $r + r^*$ is a sum of squares. The converse implication is trivial. ■

Corollary 3.10. *Let $I \subseteq \mathbb{R}\langle x, x^* \rangle$ be the left ideal generated by a single homogeneous polynomial p . Let $p = p_1 \cdots p_k$ be a factorization of p into irreducible factors. Then I is real if and only if none of the $2k$ polynomials*

$$\pm(p_1 + p_1^*), \pm(p_1 p_2 + p_2^* p_1^*), \dots, \pm(p_1 \cdots p_k + p_k^* \cdots p_1^*)$$

is a nonzero sum of squares.

For the sake of convenience, we call the highest degree homogeneous part of a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ its **leading polynomial**.

Proof. By Proposition 3.7, I is real if and only if there is no nonzero sum of squares of the form $qp + p^*q^*$, where $\deg(q) < \deg(p)$. We claim that we can restrict without loss of generality to q being homogeneous, i.e., $q = q'$. Indeed, if q' is the leading polynomial of q , then either $q'p + p^*(q')^* = 0$ or $q'p + p^*(q')^*$ is the leading polynomial of $qp + p^*q^*$. In the former case,

$$qp + p^*q^* = (q - q')p + p^*(q - q')^*,$$

so we can look at the leading polynomial of $q - q'$ and repeat. In the latter case, $q'p + p^*(q')^*$ is a sum of squares since $qp + p^*q^*$ is.

First, if $\pm(p_1 p_2 \cdots p_j + p_j^* \cdots p_2^* p_1^*)$ is a sum of squares for some $j < k$, then let $q = p_k^* p_{k-1}^* \cdots p_{j+1}^*$. We have $\deg(q) < \deg(p)$ and

$$qp + p^*q^* = p_k^* p_{k-1}^* \cdots p_{j+1}^* (\pm p_1 p_2 \cdots p_j \pm p_j^* \cdots p_2^* p_1^*) p_{j+1} \cdots p_{k-1} p_k,$$

is a sum of squares in $I + I^*$. If $\pm(p + p^*)$ is a nonzero sum of squares, then $q = 1$ works.

Conversely, suppose $qp + p^*q^*$ is homogeneous of degree $2d$, for some $d < \deg(p)$, and assume it is a nonzero sum of squares. By Lemma 3.9, $p = rq^*$ for some r such that $r + r^*$ is a sum of squares. By the uniqueness of the factorization,

$$r = \lambda p_1 \cdots p_j \quad \text{and} \quad q^* = \frac{1}{\lambda} p_{j+1} \cdots p_k,$$

for some scalar λ and some j , which implies, since $qp + p^*q^*$ is a nonzero sum of squares, that

$$\pm(p_1 \cdots p_j + p_j^* \cdots p_1^*)$$

is a nonzero sum of squares. ■

Irreducibility of the factors p_j allow a rephrasing of Corollary 3.10 which lends itself to computation.

Theorem 3.11. *Let p be a nonconstant homogeneous polynomial, and let I be the left ideal generated by p . Decompose p as $p = p_1 \cdots p_k$, where the p_i are irreducible and nonconstant. Then I is real if and only if neither of the following two hold:*

(1) *There exists j with $2j \leq k$ such that*

$$p_1 = \lambda_1 p_{2j}^*, \quad p_2 = \lambda_2 p_{2j-1}^*, \quad \dots, \quad p_j = \lambda_j p_{j+1}^*$$

for some scalars λ_i .

(2) *There exists j with $2j + 1 \leq k$ such that*

$$p_1 = \lambda_1 p_{2j+1}^*, \quad p_2 = \lambda_2 p_{2j}^*, \quad \dots, \quad p_j = \lambda_j p_{j+2}^*$$

for some scalars λ_i , and $\pm(p_{j+1} + p_{j+1}^)$ is a nonzero sum of squares.*

Remark 3.12. *This theorem implies that to test the conditions of Corollary 3.10 for $p = p_1 \cdots p_k$, one only needs to do the following. For each j with $2j \leq k$, examine whether for each i the polynomials p_i and p_{2j-i}^* are scalar multiples of each other. For each j with $2j + 1 \leq k$, examine whether for each i the polynomials p_i and p_{2j+1-i}^* are scalar multiples of each other, and if they are, check to see if $p_{j+1} + p_{j+1}^*$ is plus or minus a sum of squares. Checking if a homogeneous degree $2d$ noncommutative polynomial p is a sum of squares is straightforward: by Lemma 3.3 we only need to find its (d, d) -Gram matrix (by solving a linear system) and test whether it is positive semidefinite.*

Proof of Theorem 3.11. By Corollary 3.10, I is not real if and only if for some ℓ ,

$$\pm(p_1 \cdots p_\ell + p_\ell^* \cdots p_1^*)$$

is a nonzero sum of squares. Proceed by induction on ℓ .

If $\ell = 1$, then $\pm(p_1 + p_1^*)$ is a sum of squares and item (2) holds.

If $\ell > 1$, suppose that $\deg(p_\ell) \leq \frac{1}{2} \deg(p_1 \cdots p_\ell)$. If this were not the case, a similar argument could be made using p_1 noting that $\deg(p_\ell) > \frac{1}{2} \deg(p_1 \cdots p_\ell)$ implies that $\deg(p_1) \leq \frac{1}{2} \deg(p_1 \cdots p_\ell)$. Note that $\pm(p_1 \cdots p_\ell + p_\ell^* \cdots p_1^*)$ is a sum of homogeneous degree $\deg(p_1 \cdots p_\ell)$ polynomials and is a nonzero sum of squares. Therefore

$$\deg(p_1 \cdots p_\ell + p_\ell^* \cdots p_1^*) = \deg(p_1 \cdots p_\ell)$$

and this degree is even since $\pm(p_1 \cdots p_\ell + p_\ell^* \cdots p_1^*)$ is a sum of squares. By Lemma 3.9, this implies that p_1^* factors $p_2 \cdots p_\ell$ on the right. Since p_1 is irreducible, by uniqueness of factorization there exists some nonzero λ_1 such that $p_1 = \lambda_1 p_\ell^*$. Now either $\ell = 2$ and so item (1) holds, or $p_2 \cdots p_{\ell-1}$ satisfies

$$\pm(p_2 \cdots p_{\ell-1} + p_{\ell-1}^* \cdots p_2^*)$$

is a sum of squares. The result follows by induction. ■

Corollary 3.13. *Let $0 \neq p \in \mathbb{R}\langle x, x^* \rangle$ be homogeneous and let I be the left ideal generated by p . Then I is real if and only if p is not of the form*

$$(4) \quad p = (s + q)f,$$

where s is a nonconstant sum of squares, $q^* = -q$, and $f \in \mathbb{R}\langle x, x^* \rangle$.

Proof. Theorem 3.11 implies that I is not real if and only if one of two cases hold. In case (1), p is equal to $p = u^*uv$ for some nonconstant polynomial u . Here $s = u^*u$ and $q = 0$ gives the result. In case (2), p is equal to $p = u^*tuv$, with $t + t^*$ plus or minus a sum of squares. By multiplying v by ± 1 , we can assume without loss of generality that $t + t^*$ is a sum of squares. Setting $s = \frac{1}{2}u^*(t + t^*)u$, $q = \frac{1}{2}u^*(t - t^*)u$, and $f = v$ gives the result. The converse implication (if p is of the form (4) then I is non-real) is trivial; see Proposition 3.15 below. ■

Example 3.14. *Let $w \in \mathbb{R}\langle x, x^* \rangle$. If $w = (s + q)r$, for some nonconstant sum of squares s , some antisymmetric q , and some r , then w is a monomial if and only if $s + q$ and r are monomials, that is, $s = u^*u$ for some monomial $u \in \mathbb{R}\langle x, x^* \rangle$, $q = 0$, and r is a monomial. Therefore, Corollary 3.13 in the monomial case states that the left ideal I generated by a monomial w is real if and only if w is left unshrinkable.*

3.3. General Principal Left Ideals. To this point we have considered only homogeneous ideals. Now we consider principal left ideals with a nonhomogeneous generator. Here the results are less definitive than before.

If $p \in \mathbb{R}\langle x, x^* \rangle$ is not homogeneous but has a similar structure to that of Corollary 3.13, then the following holds.

Proposition 3.15. *If p is of the form*

$$(5) \quad p = (s + q)f,$$

where s is a nonzero sum of squares, $q^* = -q$, $f \neq 0$, and $\deg(s + q) > 0$, then the left ideal I generated by p is not real.

Proof. Since $\deg(s + q) > 0$, and $\deg(p) = \deg(f) + \deg(s + q)$, we have $\deg(f) < \deg(p)$. We see that

$$f^*p + p^*f = 2f^*sf \in I + I^*$$

is a nonzero sum of squares. By Proposition 3.7, this implies that I is not real. ■

Example 3.16. *Not every non-real principal left ideal is generated by a polynomial of the form (5). Consider*

$$p = (x_1 - x_1^* + 1)(x_2 - x_2^*) + 1.$$

One can show p is irreducible. Further,

$$p + p^* = (x_1 - x_1^*)(x_2 - x_2^*) + (x_2 - x_2^*)(x_1 - x_1^*) + 2$$

which is not a (\pm) sum of squares. However,

$$(x_2 - x_2^*)p + p^*(x_2 - x_2^*)^* = 2(x_2 - x_2^*)^*(x_2 - x_2^*)$$

is a sum of squares of elements not in the left ideal generated by p .

Proposition 3.17. *If p is of the form*

$$(6) \quad p = (s + q_1)q_2 + c,$$

where each q_i is antisymmetric, $q_2 \neq 0$, s is a nonzero sum of squares, c is a constant, and $\deg(s + q_1) > 0$ then the left ideal I generated by p is not real.

Proof. Since $\deg(s + q_1) > 0$, we see $\deg(q_2) < \deg(p)$. We see

$$q_2^*p + p^*q_2 = 2q_2^*sq_2 \in I + I^*$$

is a nonzero sum of squares, which implies that I is not real. ■

However, these propositions do not describe all polynomials generating non-real principal left ideals.

Example 3.18. *Consider the following univariate polynomial*

$$p = xx^* - (x^*)^2 + 2x + 4.$$

As is easily seen, p is not of the form (5) or (6). On the other hand, the left ideal I generated by p is non-real. Indeed,

$$(x + 2)p + p^*(x + 2)^* = (4 + 2x^*)^*(4 + 2x^*) \in I + I^*,$$

but $4 + 2x^* \notin I$. We shall investigate reality of ideals generated by quadratics in more detail in Subsection 4.4 below.

Proposition 3.19. *Let I be the left ideal generated by some polynomial p . Let p' be the leading polynomial of p , and let $p' = p_1 \cdots p_k$ be a factorization of p' into irreducible parts. If I is not real, then at least one of*

$$\pm(p_1 + p_1^*), \pm(p_1p_2 + p_2^*p_1^*), \dots, \pm(p_1 \cdots p_k + p_k^* \cdots p_1^*)$$

is a sum of squares.

Proof. By Proposition 3.7, if I is not real, then there exists a q with $\deg(q) < \deg(p)$ and $qp + p^*q^*$ being a nonzero sum of squares. Let q' be the leading polynomial of q . Then

$$q'p' + (p')^*(q')^*$$

is either 0 or the leading polynomial of $qp + p^*q^*$, in which case it is a sum of squares. Hence, in either case, $q'p' + (p')^*(q')^*$ is a sum of squares. By Lemma 3.9, this necessarily implies that $p' = p_1 \cdots p_j(\lambda q')$ for some j and some nonzero λ , and that $\pm(p_1 \cdots p_j + p_j^* \cdots p_1^*)$ is a sum of squares. ■

3.4. Further Examples. Recall that a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ is analytic if it contains no x_j^* , i.e., if $p \in \mathbb{R}\langle x \rangle$. Likewise, a polynomial $p \in \mathbb{R}\langle x^* \rangle$ is called **antianalytic**.

Lemma 3.20. *Let $a, b \in \mathbb{R}\langle x, x^* \rangle$ be homogeneous analytic polynomials such that $\deg(a) = \deg(b) > 0$. Then $a + b^*$ is irreducible.*

Proof. Assume

$$a + b^* = \left(\sum_{\substack{u \in \langle x, x^* \rangle \\ \deg(u)=d}} A_u u \right) \left(\sum_{\substack{w \in \langle x, x^* \rangle \\ \deg(w)=e}} B_w w \right),$$

for some $d, e \in \mathbb{N}$. The coefficient of uw in $a + b^*$ is $A_u B_w$. We see that $a + b^*$ has only terms which are analytic or antianalytic. Suppose u_0, w_0 are such that $u_0 w_0$ is analytic. Then u_0, w_0 are analytic. This implies that for any other u , either $A_u = 0$ or uw_0 is analytic or antianalytic, which necessarily implies that u is analytic. Similarly, either $B_w = 0$ or w is analytic. This implies that $a + b^*$ is analytic, which is a contradiction. ■

Proposition 3.21. *Suppose $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is the left ideal generated by a nonconstant polynomial $p = a + b^*$, where a and b are analytic polynomials. Then I is not real if and only if $p = a - a^* + c$ for some nonzero constant c .*

Proof. First, if $p = a - a^* + c$, then $p + p^* = 2c \in I + I^*$, which implies that $1 \in \sqrt{I}$. Therefore I is not real.

Conversely, suppose I is not real. Let $d = \deg(p)$. We have $\deg(p) = \max\{\deg(a), \deg(b)\}$ since the leading polynomials of a and b^* are analytic and antianalytic respectively, and hence cannot cancel each other out. Let a', b' be the degree d terms of a and b respectively, so that $p' = a' + (b')^*$ is the leading polynomial of p .

If $b' = 0$, then $p' = a'$. By Proposition 3.19, $p' = p_1 \cdots p_j f$ for some f and some nonconstant irreducible factors p_i , and

$$\pm(p_1 \cdots p_j + p_j^* \cdots p_1^*)$$

is a sum of squares. However, since p' is analytic, this implies that $p_1 \cdots p_j$ is analytic. An analytic polynomial has no terms of the form m^*m , whereas a nonconstant sum of squares does. This is a contradiction, so $b' \neq 0$. Similar reasoning shows $a' \neq 0$.

Next, by Lemma 3.20, $a' + (b')^*$ is irreducible. By Proposition 3.19,

$$a' + b' + (a')^* + (b')^*$$

is a sum of squares. A sum of an analytic and an antianalytic polynomial has no terms of the form m^*m , whereas a nonconstant sum of squares does. Therefore $a' + b' + (a')^* + (b')^* = 0$, which implies that $a' = -b'$.

Next, since I is not real, there exists a q such that $\deg(q) < \deg(p)$ and $qp + p^*q^*$ is a nonzero sum of squares. Let q' be the leading polynomial of q . Then

$$q'(a' - (a')^*) + (a' - (a')^*)^*(q')^*$$

is either 0 or is the leading polynomial of $qp + p^*q^*$. In either case, it is a sum of squares. Since $\deg(q) < \deg(p) = \deg(a' - (a')^*)$, by Lemma 3.9, this implies that $(q')^*$ is a factor of $a' - (a')^*$ on the right. However, by Lemma 3.20, $a - a^*$ is irreducible. Therefore q' is constant, which implies that q is constant. Therefore $\pm(p + p^*)$ is a nonzero sum of squares.

Let $e = \deg(p + p^*)$. Let a'_e, b'_e be the degree e elements of a and b respectively so that $a'_e + (b'_e)^*$ is the leading polynomial of $p + p^*$. As before, a nonconstant sum of squares cannot be expressed in this form. Hence $e = 0$, and so p is of the form $a - a^* + c$, where $c = b_e$ is a nonzero constant. ■

Example 3.22. Proposition 3.21 only applies to principal left ideals. Indeed, if I is generated by x_1^2 and $x_1 + x_1^*$, then I is not real. We have $-x_1^2 + (x_1)(x_1 + x_1^*) = x_1x_1^* \in I + I^*$ but $x_1 \notin I$.

Corollary 3.23. Suppose $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is the left ideal generated by a polynomial p with $\deg(p) = 1$. Then I is real if and only if p is not of the form $a - a^* + c$, where a is analytic and homogeneous of degree 1 and $c \neq 0$ is a constant.

Proof. Every polynomial p of degree 1 is of the form $p = a + b^*$, where a, b are analytic, and a has no constant term. The result follows from Proposition 3.21. ■

4. ALGORITHM FOR CHECKING WHETHER $I = \sqrt{I}$

The paper [CHMN+] gives an algorithm for computing the real radical of any finitely generated left ideal I . Here we discuss an improvement of a more limited algorithm which determines whether or not a given left ideal I is real. A test version has been implemented, and we give examples. Also we shall give a Gröbner basis algorithm for left ideals and a

theorem to the effect that it is well behaved, contrary to what one sees with Gröbner bases for two sided ideals. The left Gröbner basis algorithm is useful though not essential for our real ideal algorithm.

4.1. An algorithm for determining if a principal ideal is real. Suppose $p \in \mathbb{R}\langle x, x^* \rangle_d$ is given and let I be the left ideal generated by p . We employ Proposition 3.7 to test reality of I . Consider the feasibility problem

$$(7) \quad \begin{aligned} & s \in \mathbb{R}\langle x, x^* \rangle \quad \text{is a nonzero sum of squares} \\ \text{s.t. } & s = qp + p^*q^* \quad \text{for some } q \text{ with } \deg(q) < \deg(p). \end{aligned}$$

This is an instance of an LMI. Namely, let $M_{<d}$ be a vector whose entries are all monomials of degree $< d$. Then s is a sum of squares if and only if there is a $G \succeq 0$ with $M_{<d}^* G M_{<d} = s$; cf. [MP05, KP10, Hel02]. The equation $s = qp + p^*q^*$ translates into a system of linear equations involving the coefficients of q and entries of G . Thus (7) is a feasibility semidefinite program (SDP)

$$(8) \quad \begin{aligned} & \text{Find } 0 \neq G \succeq 0 \\ \text{s.t. } & M_{<d}^* G M_{<d} = qp + p^*q^*, \quad \deg(q) < d \end{aligned}$$

and can thus be solved using a standard SDP solver. To search for a nonzero G we normalize (8) by requiring $\text{tr}(G) = 1$. The left ideal I is non-real if and only if (8) is feasible.

Remark 4.1. *We remark that checking if a noncommutative polynomial is a sum of squares can be done exactly using quantifier elimination [BPR06], but this is only viable for problems of small size (since the complexity for quantifier elimination is doubly exponential [BPR06, Section 11]). Hence in practice we employ SDPs (which typically run in polynomial time, cf. [WSV00]) for numerical verification; cf. NCS0Stools [CKP11] for a computer algebra package which does this.*

We demonstrate the above with a simple example.

Example 4.2. *Let $p = xx^* - x^*x - 1$. The corresponding left ideal I is real. Indeed, write $q = q_0 + q_1x + q_2x^*$, and $G = [g_{ij}]_{i,j=1}^3$. Then (8) becomes*

$$G \succeq 0,$$

$$\begin{aligned}
(9) \quad & g_{11} + g_{22} + g_{33} - 1 = 0 \\
& g_{11} + 2q_0 = 0 \\
& g_{11} - 2g_{12} - 2g_{13} + g_{22} + 2g_{23} + 2g_{33} + 2q_1 = 0 \\
& g_{11} + g_{12} + g_{13} - g_{22} - 3g_{23} - 2g_{33} + 4q_0 - q_1 - 3q_2 = 0 \\
& g_{11} - 4g_{12} - 4g_{13} + 5g_{22} + 8g_{23} + 4g_{33} + 4q_0 - 8q_1 - 6q_2 = 0 \\
& g_{11} + 2g_{12} + 2g_{13} + 2g_{22} + 2g_{23} + 2g_{33} + 2q_0 + 2q_1 + 2q_2 = 0 \\
& g_{11} - 2g_{12} - 2g_{13} + g_{22} + 2g_{23} + g_{33} + 2q_0 - 2q_1 - 2q_2 = 0 \\
& g_{11} + g_{22} + 4q_0 + 2q_2 = 0
\end{aligned}$$

Solving (9) yields

(10)

$$g_{11} = 1, \quad g_{13} = -g_{12}, \quad g_{22} = 1, \quad g_{23} = 0, \quad g_{33} = -1, \quad q_0 = -\frac{1}{2}, \quad q_1 = 0, \quad q_2 = 0.$$

Putting (10) back into G leads to the LMI

$$G = \begin{bmatrix} 1 & g_{12} & -g_{12} \\ g_{12} & 1 & 0 \\ -g_{12} & 0 & -1 \end{bmatrix} \succeq 0,$$

which is clearly infeasible.

4.2. Left Gröbner Basis Algorithm on $\mathbb{R}\langle x, x^* \rangle$. A useful tool for the general algorithm for testing reality of non-principal left ideals is computation of a left Gröbner basis for an ideal. This was implicit in [CHMN+] but here we give a cleaner description which lends itself to our computation. For details we refer the reader to the extensive literature on noncommutative Gröbner basis; see e.g. [Mor94, Rei95, Kel97, Hey01, Lev05] and the references therein.

Fix a monomial order \succ on $\langle x, x^* \rangle$. For convenience, we choose an order such that $\deg(u) < \deg(v)$ implies $u \prec v$. By definition, \succ satisfies the descending chain condition.

Let p_1, \dots, p_k generate a left ideal $I \subseteq \mathbb{R}\langle x, x^* \rangle$, and assume that they are monic (i.e. the coefficient of the leading monomial is 1). Let u_1, \dots, u_k be the leading monomials of p_1, \dots, p_k respectively. If $u_i \mid u_j$ for any $i \neq j$, let ω be a monomial such that $u_i = \omega u_j$. Replace p_j by $A(p_j - \omega p_i)$, where A is a normalization making the latter polynomial monic. In this case, the leading monomial of this new p_j is lower than the leading monomial of the old p_j . If any of the new p_j are 0, remove them from our set.

Repeat this until $u_i \nmid u_j$ for any $i \neq j$. Then $\{p_1, \dots, p_k\}$ is a left Gröbner basis. This Algorithm is guaranteed to terminate since at each step where the algorithm does not stop, we replace a polynomial with another polynomial whose leading monomial is lower than that of the polynomial being replaced.

Proposition 4.3. *Let p_1, \dots, p_k be a left Gröbner basis for a left ideal I , and suppose $\deg(p_i) \leq d$ for each i . Then for each $e \geq d$, a basis for I_e is*

$$(11) \quad \{vp_i \mid 1 \leq i \leq k, v \in \langle x, x^* \rangle_{e-\deg(p_i)}\}.$$

Proof. Any polynomial $\iota \in I$ is equal to

$$\iota = q_1p_1 + \dots + q_kp_k$$

for some polynomials q_i since the p_i generate I . Let ω_i be the leading monomial of each q_i , and let u_i be the leading monomial of each p_i so that the leading monomial of $q_i p_i$ is $\omega_i u_i$.

If $i \neq j$, then $\omega_i u_i \neq \omega_j u_j$; indeed, suppose $\omega_i u_i = \omega_j u_j$, and suppose $\deg(u_i) \leq \deg(u_j)$. Then $\omega_i u_i = \omega_j u_j$ implies that $u_i \mid u_j$, which is a contradiction by the construction of the left Gröbner basis.

Therefore none of the leading monomials of the $q_i p_i$ cancel each other out. In particular, $\deg(\iota) = \max\{\deg(q_i p_i)\}$. This shows that the set (11) spans I_e for each $e \geq d$. Likewise, since the leading monomials of $v_i p_i$ cannot cancel each other out, the set (11) is linearly independent. \blacksquare

4.3. The Real Ideal Algorithm for finitely generated left ideals. Suppose $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is a finitely generated left ideal with left Gröbner basis $\{p_1, \dots, p_r\} \subseteq \mathbb{R}\langle x, x^* \rangle_d$. Consider the feasibility problem

$$(12) \quad \begin{aligned} & s \in \mathbb{R}\langle x, x^* \rangle \quad \text{is a nonzero sum of squares} \\ \text{s.t. } & s = \sum_{j=1}^r (q_j p_j + p_j^* q_j^*), \quad \deg(q_j p_j) < 2d. \end{aligned}$$

As in Subsection 4.1, this is an instance of an LMI,

$$(13) \quad \begin{aligned} & \text{Find } 0 \neq G \succeq 0 \\ \text{s.t. } & M_{<d}^* G M_{<d} = \sum_{j=1}^r (q_j p_j + p_j^* q_j^*), \quad \deg(q_j p_j) < 2d. \end{aligned}$$

and can thus be solved using a standard SDP solver. To search for a nonzero G we normalize (13) by requiring $\text{tr}(G) = 1$. The ideal I is real if and only if (13) is infeasible.

Example 4.4. *Here is a simple univariate example. Let \succ be the following monomial order on $\langle x, x^* \rangle$: for $u, w \in \langle x, x^* \rangle$, we define $u \prec w$ if $\deg(u) < \deg(w)$ or if $\deg(u) = \deg(w)$ and $u = rx^*s$ and $w = rxt$ for some $r, s, t \in \langle x, x^* \rangle$.*

Let the following set generate a left ideal

$$S = \{x^3 + 1, x^2 + (x^*)^2, xx^* - (x^*)^2, x^*x - 5\}.$$

Observe that these polynomials are presented in an \succ -decreasing manner. We see x^2 divides x^3 , so we replace $x^3 + 1$ in S with

$$-(x^3 + 1 - x(x^2 + (x^*)^2)) = x(x^*)^2 - 1.$$

We thus obtain the generating set

$$(14) \quad S' = \{x(x^*)^2 - 1, x^2 + (x^*)^2, xx^* - (x^*)^2, x^*x - 5\}.$$

Clearly, S' is a left Gröbner basis. We used a Mathematica implementation of (13) based on NCAlgebra [HOSM12] to verify that the left ideal generated by S' is real.

To construct (12) or (13), any generators p_j for I will do, the fewer the better. Finding a smallest generating set for an ideal is hard, hence a reduced Gröbner basis is a reasonable choice. The left Gröbner basis algorithm is useful in producing a fairly small basis for the ideal I .

We have implemented the algorithm for test purposes. A major limitation is construction of the LMI required. This arises from manipulation of the large number of monomials in $M_{<d}^*GM_{<d}$. One could improve performance by storing these in advance. Also, the benefits when the p_j have few terms could be explored further.

4.4. Principal ideals generated by univariate quadratic noncommutative polynomials. In Subsection 3.4 we characterized linear polynomials giving rise to real principal left ideals. In this section we discuss a complete classification whether or not the left ideal generated by a univariate quadratic polynomial is real. It seems that there is no clean and neat closed form solution, so our presentation gives a rather lengthy case by case analysis. We start by characterizing univariate quadratics which are sums of squares.

Proposition 4.5. *For a symmetric univariate quadratic polynomial*

$$p = a_0 + a_1(x + x^*) + a_2(x^2 + (x^*)^2) + a_3xx^* + a_4x^*x,$$

the following are equivalent:

- (i) p is a sum of two squares;
- (ii) p is a sum of squares;
- (iii) the following LMI is feasible

$$(15) \quad G = \begin{bmatrix} a_0 & \lambda & a_1 - \lambda \\ \lambda & a_3 & a_2 \\ a_1 - \lambda & a_2 & a_4 \end{bmatrix} \succeq 0;$$

- (iv) $p(X) \succeq 0$ for all $n \in \mathbb{N}$ and all $X \in \mathbb{R}^{n \times n}$;
- (v) $p(X) \succeq 0$ for all $X \in \mathbb{R}^{2 \times 2}$;

(vi) *the matrix*

$$(16) \quad \begin{bmatrix} a_3 & a_2 \\ a_2 & a_4 \end{bmatrix}$$

is positive semidefinite, $-a_1^2 + a_0(2a_2 + a_3 + a_4) \geq 0$, and $a_0 \geq 0$.

Proof. It is clear that (i) implies (ii). By noting that G in (15) is the general form for the Gram matrix of p , we can deduce that items (ii) and (iii) are equivalent. Now suppose (iii) holds. By choosing the smallest λ making $G = G(\lambda) \succeq 0$, the rank of the corresponding G is ≤ 2 . As in the proof of Lemma 3.3 this implies p is a sum of ≤ 2 squares, so (i) holds. By the sum of squares theorem [Hel02], (ii) and (iv) are equivalent. All this shows that (i), (ii), (iii) and (iv) are equivalent.

Clearly, (iv) implies (v). Let us now establish (v) \Rightarrow (vi). First of all, $a_0 = p(0) \geq 0$. Since p is positive semidefinite on $\mathbb{R}^{2 \times 2}$, its homogeneous degree 2 part, \hat{p} , is also positive semidefinite on 2×2 matrices. It follows that

$$(17) \quad \text{tr} \left(\hat{p} \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \right) = a_3 + a_4 \geq 0,$$

and

$$(18) \quad \det \left(\hat{p} \left(\begin{bmatrix} 0 & 1 \\ 0 & c \end{bmatrix} \right) \right) = a_3 a_4 + c^2 (a_3 a_4 - a_2^2) \geq 0 \quad \text{for all } c \in \mathbb{R}.$$

From (18) we immediately obtain

$$(19) \quad a_3 a_4 - a_2^2 \geq 0.$$

Together (17) and (19) show that the matrix (16) has nonnegative trace and nonnegative determinant, so is positive semidefinite. In particular,

$$2a_2 + a_3 + a_4 = \left\langle \begin{bmatrix} a_3 & a_2 \\ a_2 & a_4 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle \geq 0.$$

If $2a_2 + a_3 + a_4 = 0$, then $p(c) = a_0 + 2ca_1 \geq 0$ for all $c \in \mathbb{R}$, whence $a_1 = 0$. In this case $-a_1^2 + a_0(2a_2 + a_3 + a_4) = 0$. If $2a_2 + a_3 + a_4 > 0$, then

$$p \left(-\frac{a_1}{2a_2 + a_3 + a_4} \right) = \frac{-a_1^2 + a_0(2a_2 + a_3 + a_4)}{2a_2 + a_3 + a_4} \geq 0$$

shows $-a_1^2 + a_0(2a_2 + a_3 + a_4) \geq 0$, as desired.

Finally, assume (vi) holds. If $2a_2 + a_3 + a_4 = 0$, then $a_1 = 0$, and hence p has no linear terms. Since (16) is positive semidefinite, the homogeneous quadratic part of p is a sum of

squares. As also $a_0 \geq 0$, p is a sum of squares. We can thus assume $2a_2 + a_3 + a_4 > 0$. Performing an affine linear change of variables $x \mapsto x - \frac{a_1}{2a_2 + a_3 + a_4}$ leads to the polynomial

$$\tilde{p}(x) = p\left(x - \frac{a_1}{2a_2 + a_3 + a_4}\right) = \frac{-a_1^2 + a_0(2a_2 + a_3 + a_4)}{2a_2 + a_3 + a_4} + a_2(x^2 + (x^*)^2) + a_3xx^* + a_4x^*x$$

having no linear term. Obviously, p is positive (resp., sum of squares) iff \tilde{p} is positive (resp., sum of squares), and the latter is the case iff its constant term is nonnegative and its homogeneous quadratic part is a sum of squares, i.e., iff $-a_1^2 + a_0(2a_2 + a_3 + a_4) \geq 0$ and (16) is positive semidefinite. ■

Remark 4.6. *The quantitative strengthening of the sum of squares theorem [Hel02] (cf. [MP05, McC01]) tells us that a p as in Proposition 4.5 is a sum of squares iff it is positive semidefinite for all $X \in \mathbb{R}^{3 \times 3}$. Item (v) above improves this size bound a bit.*

Let

$$(20) \quad p = a_0 + a_1x + a_2x^* + a_3x^2 + a_4xx^* + a_5x^*x + a_6(x^*)^2 \in \mathbb{R}\langle x, x^* \rangle$$

denote an arbitrary univariate and quadratic, and let I be the principal left ideal it generates. We now set about to determine if I is real.

By Proposition 3.7, I is non-real if and only if there is a linear

$$(21) \quad q = q_0 + q_1x + q_2x^* \in \mathbb{R}\langle x, x^* \rangle$$

such that $qp + p^*q^*$ is a nonzero sum of squares. Compute $qp + p^*q^*$ and verify that it equals

$$(22) \quad \begin{aligned} & 2a_0q_0 + x(a_0q_1 + a_0q_2 + a_1q_0 + a_2q_0) + x^*(a_0q_1 + a_0q_2 + a_1q_0 + a_2q_0) \\ & + x^2(a_1q_1 + a_2q_2 + a_3q_0 + a_6q_0) + (x^*)^2(a_1q_1 + a_2q_2 + a_3q_0 + a_6q_0) \\ & + 2x^*x(a_1q_2 + a_5q_0) + 2xx^*(a_2q_1 + a_4q_0) \\ & + x^3(a_3q_1 + a_6q_2) + (x^*)^3(a_3q_1 + a_6q_2) + x^2x^*(a_6q_1 + a_4q_1) + x(x^*)^2(a_6q_1 + a_4q_1) \\ & + xx^*x(a_4q_2 + a_5q_1) + x^*x^2(a_3q_2 + a_5q_2) + (x^*)^2x(a_3q_2 + a_5q_2) + x^*xx^*(a_4q_2 + a_5q_1). \end{aligned}$$

Here line 1 of (22) collects degree 0 and 1 terms of $qp + p^*q^*$, lines 2 and 3 contain degree 2 terms, while lines 4 and 5 contain degree 3 terms. If (22) is a sum of squares, all degree 3 terms must vanish (and we are thrown into the case of Proposition 4.5).

All degree 3 terms vanishing is equivalent to following system of equations:

$$(23) \quad \begin{aligned} & a_3q_1 + a_6q_2 = 0 \\ & (a_4 + a_6)q_1 = 0 \\ & a_4q_2 + a_5q_1 = 0 \\ & (a_3 + a_5)q_2 = 0. \end{aligned}$$

We now perform a case by case analysis. Note that I is real whenever (23) does not have a solution. Thus we turn our attention to solutions of (23).

- (a) If $a_4 + a_6 \neq 0$ and $a_3 + a_5 \neq 0$, then q is a constant. In this case I is non-real iff
- $$p + p^* = 2a_0 + (a_1 + a_2)x + (a_1 + a_2)x^* + (a_3 + a_6)x^2 + 2a_4xx^* + 2a_5x^*x + (a_3 + a_6)(x^*)^2,$$
- is a \pm sum of squares. By Proposition 4.5, this can be reformulated as the disjunction of two systems of polynomial inequalities.
- (b) Suppose $a_3 + a_5 \neq 0 = a_4 + a_6$. Then $q_2 = 0$, and the polynomial system (23) simplifies considerably: $a_3q_1 = 0$, $a_5q_1 = 0$. By $a_3 + a_5 \neq 0$, this implies $q_1 = 0$, i.e., q is constant. We now proceed as in case (a).
- (c) The case $a_3 + a_5 = 0 \neq a_4 + a_6$ is handled similarly. Alternately, use the automorphism of the free algebra sending $x \mapsto x^*$ to reduce to case (b).
- (d) Assume $a_3 + a_5 = 0 = a_4 + a_6$. Then $a_5 = -a_3$ and $a_6 = -a_4$, and the system (23) reduces to the single equation

$$-a_3q_1 + a_4q_2 = 0.$$

If $a_3 = a_4 = 0$, then p is linear and then Corollary 3.23 tells us when I is real. Otherwise $a_3 \neq 0$ or $a_4 \neq 0$. We consider the case $a_4 \neq 0$, and note that the former can be reduced to the latter via $x \mapsto x^*$.

Solving (23) leads to $q_2 = \frac{a_3}{a_4}q_1$. We reevaluate $qp + p^*q^*$:

$$(24) \quad 2a_0q_0 + x\left(a_1q_0 + a_2q_0 + a_0q_1 + \frac{a_0a_3q_1}{a_4}\right) + x^*\left(a_1q_0 + a_2q_0 + a_0q_1 + \frac{a_0a_3q_1}{a_4}\right) \\ + x^2\left(a_3q_0 - a_4q_0 + a_1q_1 + \frac{a_2a_3q_1}{a_4}\right) + (x^*)^2\left(a_3q_0 - a_4q_0 + a_1q_1 + \frac{a_2a_3q_1}{a_4}\right) \\ + 2xx^*(a_4q_0 + a_2q_1) + 2x^*x\left(-a_3q_0 + \frac{a_1a_3q_1}{a_4}\right)$$

Then I is non-real if and only if there are $q_0, q_1 \in \mathbb{R}$ making (24) a nonzero sum of squares. By assertion (vi) of Proposition 4.5, (24) is a sum of squares if and only if the system of the following polynomials inequalities is feasible.

$$(25) \quad \begin{aligned} a_4q_0 + a_2q_1 &\geq 0 \\ a_4(a_3 + a_4)q_0 &= (a_1a_4 - a_2a_3)q_1 \\ (a_1 + a_2)a_4q_0 &= a_0(a_3 + a_4)q_1 \\ a_0q_0 &\geq 0 \\ a_3a_4(a_1q_1 - a_4q_0) &\geq 0 \end{aligned}$$

We still assume that $a_4 \neq 0$ and we also require that (24) has at least one nonzero coefficient.

If the linear system (for q_0, q_1) of the second and the third equation in (25) has a nonzero determinant, then $q_0 = q_1 = 0$ which implies a contradiction that all coefficients of (24) are zero. Therefore

$$(26) \quad (a_1 + a_2)(a_1a_4 - a_2a_3) = a_0(a_3 + a_4)^2.$$

If $a_3 + a_4 = 0$, (25) simplifies considerably. We can deduce from the second and the third equation that $a_1 + a_2 = 0$. The inequality in the last line of (25) is now equivalent to the first, so we are left with the system of the first and the fourth inequality. Moreover, (24) has a nonzero coefficient iff $a_4q_0 + a_2q_1 \neq 0$ or $a_0q_0 \neq 0$. This system has a solution iff either $a_2 \neq 0$ or $a_0a_4 \geq 0$. In the first case we can take $q_0 = 0$ and $q_1 = a_2$; in the second $q_0 = a_4$ and $q_1 = 0$.

If $a_3 + a_4 \neq 0$ we can compute a_0 from (26) and q_0 from the second equation of (25). In this case the system (25) is equivalent to $\frac{(a_1 + a_2)a_4q_1}{a_3 + a_4} \geq 0$. Moreover, (24) has a nonzero coefficient iff $(a_1 + a_2)q_1 \neq 0$. This system has a solution iff $a_1 + a_2 \neq 0$. In this case we can take $q_0 = (a_1 + a_2)(a_1a_4 - a_2a_3)$ and $q_1 = (a_1 + a_2)(a_3 + a_4)a_4$.

Therefore, in case (d) (assuming $a_4 \neq 0$), the ideal I is non-real iff either

$$a_3 + a_4 = 0 \quad \text{and} \quad a_1 + a_2 = 0 \quad \text{and} \quad (a_2 \neq 0 \text{ or } a_0a_4 \geq 0)$$

or

$$a_4 + a_3 \neq 0 \quad \text{and} \quad a_1 + a_2 \neq 0 \quad \text{and} \quad a_0 = \frac{(a_1 + a_2)(a_1a_4 - a_2a_3)}{(a_3 + a_4)^2}.$$

5. NON-FINITELY-GENERATED LEFT IDEALS

One can extend Theorem 1.2 to non-finitely-generated left ideals by noticing the following: if $I = \bigcap_{\alpha} I_{\alpha}$, and each of the I_{α} is finitely generated and real, then

$$\sqrt{I} \subseteq \bigcap_{\alpha} \sqrt{I_{\alpha}} = \bigcap_{\alpha} I_{\alpha} = I \subseteq \sqrt{I}.$$

Therefore, to show that a left ideal I satisfies $I = \sqrt{I}$, it suffices to show it is an intersection of finitely-generated real left ideals.

Proposition 5.1. *Suppose I is generated by homogeneous analytic polynomials p_1, \dots, p_k, \dots . Then $\sqrt{I} = I$.*

Proof. For each $d \in \mathbb{N}$, let $I^{(d)}$ be the ideal generated by all polynomials p_i of degree $\leq d$ as well as all analytic words of degree $d+1$. In this case, each $p_j \in I^{(d)}$, and each $I^{(d)}$ is finitely generated. Further, since $I^{(d)}$ is generated by analytic polynomials, it is real. Therefore

$$\sqrt{I} \subseteq \bigcap_{d=1}^{\infty} I^{(d)}.$$

At the same time

$$I = \bigcap_{d=1}^{\infty} I^{(d)},$$

since the elements of $I^{(e)}$ of degree bounded by d , where $e \geq d$, are precisely the elements of I of degree bounded by d , cf. Lemma 2.1 ■

The next case we treat has infinitely many left unshrinkable words as generators.

Proposition 5.2. *Let $I \subseteq \mathbb{R}\langle x, x^* \rangle$ be the left ideal generated by the polynomials*

$$x_1, x_1 x_2^* x_2, \dots, x_1 (x_2^* x_2)^d, \dots$$

Then $\sqrt{I} = I$.

Proof. Let I_λ be the left ideal generated by x_1 and $x_2^* x_2 - \lambda$, where $\lambda > 0$.

Step 1: I_λ Is Real. Since I_λ is generated by polynomials of degree bounded by 2, we only have to check that no sum of squares of polynomials outside of I of degree 1 or less is in $I_\lambda + I_\lambda^*$. Since $x_1 \in I_\lambda$, we need only consider squares of polynomials in the span of all monomials of degree 1 or less which are not equal to I . Suppose that

$$p = \begin{bmatrix} x_1^* \\ x_2 \\ x_2^* \\ 1 \end{bmatrix}^* A \begin{bmatrix} x_1^* \\ x_2 \\ x_2^* \\ 1 \end{bmatrix} \in (I_\lambda + I_\lambda^*)$$

is a symmetric polynomial, i.e. A is symmetric. It is straightforward to show that p must be in the span of $x_2^* x_2 - \lambda$ and $x_1 + x_1^*$. Therefore,

$$A = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & -\lambda\alpha \end{bmatrix}.$$

If p is a nonzero sum of squares, then $(0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2$, which implies that

$$\beta = 0, \alpha \neq 0, \quad \text{and} \quad \alpha \begin{bmatrix} 1 & 0 \\ 0 & -\lambda 1 \end{bmatrix} \succeq 0.$$

But this contradicts $\lambda > 0$.

Step 2: $\bigcap_{\lambda>0} I_\lambda = I$. First, consider

$$x_1(x_2^*x_2)^d.$$

If $d = 0$, then $x_1 \in I_\lambda$. Further, we see that

$$x_1(x_2^*x_2)^d = \lambda x_1(x_2^*x_2)^{d-1} + x_1(x_2^*x_2)^{d-1}(x_2^*x_2 - \lambda),$$

hence by induction $x_1(x_2^*x_2)^d \in I_\lambda$ for all d . Since I_λ contains the generators of I ,

$$I \subseteq \bigcap_{\lambda>0} I_\lambda.$$

Next, suppose $p \in \bigcap_{\lambda>0} I_\lambda$. First, $p \in I_1$, so p is of the form

$$p = qx_1 + \sum_{i=0}^d r_i(x_2^*x_2 - 1)^i,$$

for some d , where $r_i, q \in \mathbb{R}\langle x, x^* \rangle$. Without loss of generality, assume that the terms of the r_i are not divisible on the right by $(x_2^*x_2)$; for example, if $r_i = \tilde{r}_i x_2^*x_2$ for some \tilde{r}_i , then

$$r_i(x_2^*x_2 - 1)^i = \tilde{r}_i(x_2^*x_2)(x_2^*x_2 - 1)^i = \tilde{r}_i(x_2^*x_2 - 1)^{i+1} + \tilde{r}_i(x_2^*x_2 - 1)^i.$$

Given another $\lambda > 0$, $p \in I_\lambda$. We see that

$$x_2^*x_2 - 1 + I_\lambda = \lambda - 1 + I_\lambda,$$

and inductively that

$$(x_2^*x_2 - 1)^i + I_\lambda = (\lambda - 1)^i + I_\lambda.$$

Therefore

$$\sum_{i=0}^d r_i(\lambda - 1)^i \in I_\lambda.$$

The leading term of this resulting polynomial must be divisible on the right by x_1 or $x_2^*x_2$ since

$$\{x_1, x_2^*x_2 - \lambda\}$$

is a left Grobner basis. Since the terms of each r_i are not divisible on the right by $x_2^*x_2$ by construction, the leading term of $\sum_{i=0}^d r_i(\lambda - 1)^i$ is divisible on the right by x_1 . Since this is true for arbitrary $\lambda > 0$, the leading term of some r_i is divisible on the right by x_1 . Let this term be denoted r'_i . Then

$$p - r'_i(x_2^*x_2)^i \in \bigcap_{\lambda>0} I_\lambda$$

since $r'_i(x_2^*x_2)^i \in I \subseteq \bigcap_{\lambda>0} I_\lambda$ and is in I if and only if p is. We can reduce p inductively to deduce that $p \in I$. ■

Note that in the preceding proof, I was a monomial ideal, but we rewrote it as the intersection of finitely-generated ideals which were not even generated by homogeneous polynomials.

6. STRICTLY POSITIVE LINEAR FUNCTIONALS AND *-REPRESENTATIONS

In this section we describe something of a dual to our considerations so far. For a left ideal $J \subseteq \mathbb{R}\langle x, x^* \rangle$ we call a linear functional $L : \mathbb{R}\langle x, x^* \rangle \rightarrow \mathbb{R}$ a **separating functional** for J provided

- (1) $L(J) = \{0\}$;
- (2) $L(p^*p) \geq 0$ for all $p \in \mathbb{R}\langle x, x^* \rangle$;
- (3) $L(p^*) = L(p)$ for all $p \in \mathbb{R}\langle x, x^* \rangle$;
- (4) $L(p^*p) > 0$ for $p \notin J$.

Once such a linear functional exists one can prove results of the type used in this paper to produce our examples. We do not describe this here and instead refer the reader to [CHMN+].

In this section we show how to construct a separating functional for real left monomial ideals, since this is elegant and follows a very different path than the constructions presented in [CHMN+], and actually is more general than previously proved results since this applies to left monomial ideals which are not finitely generated.

Proposition 6.1. *Suppose the left monomial ideal $J \subseteq \mathbb{R}\langle x, x^* \rangle$ is real. Then there exists a separating linear functional $L : \mathbb{R}\langle x, x^* \rangle \rightarrow \mathbb{R}$ for J .*

Proof. The proof mimics [MP05, Lemma 3.2] in that it proceeds inductively to construct L .

Start by defining $L(1) = 1$. Now suppose L_d is already defined on $\mathbb{R}\langle x, x^* \rangle_{2d}$ and satisfies the required separativity properties up to degree $2d$. We define an extension $L_{d+1} : \mathbb{R}\langle x, x^* \rangle_{2d+2} \rightarrow \mathbb{R}$ of L_d : all words of degree $2d+1$ and all words of degree $2d+2$ that are not hermitian squares, are sent to 0. For a word $w = u^*u$ of degree $2d+2$, we set

$$L_{d+1}(w) = \begin{cases} c & | w \notin J \\ 0 & | w \in J \end{cases}$$

for a constant $c \in \mathbb{R}_{>0}$ to be chosen later.

Consider a hermitian square

$$\left(\sum_{w \in \langle x, x^* \rangle_{d+1}} a_w w \right)^* \left(\sum_{w \in \langle x, x^* \rangle_{d+1}} a_w w \right).$$

One can decompose this as

$$\begin{aligned}
 (27) \quad \left(\sum_{w \in \langle x, x^* \rangle_{d+1}} a_w w \right)^* \left(\sum_{w \in \langle x, x^* \rangle_{d+1}} a_w w \right) &= \left(\sum_{w \in J} a_w w + \sum_{u \notin J} a_u u \right)^* \left(\sum_{w \in J} a_w w + \sum_{u \notin J} a_u u \right) = \\
 &= \left(\sum_{w \in J} a_w w \right)^* \left(\sum_{w \in J} a_w w \right) + \left(\sum_{w \in J} a_w w \right)^* \left(\sum_{u \notin J} a_u u \right) \\
 &\quad + \left(\sum_{u \notin J} a_u u \right)^* \left(\sum_{w \in J} a_w w \right) + \left(\sum_{u \notin J} a_u u \right)^* \left(\sum_{u \notin J} a_u u \right)
 \end{aligned}$$

The first, second and third term of the right-hand side of (27) are in $J + J^*$, and therefore are mapped by L to 0, which implies that

$$L \left(\left(\sum_{w \in \langle x, x^* \rangle_{d+1}} a_w w \right)^* \left(\sum_{w \in \langle x, x^* \rangle_{d+1}} a_w w \right) \right) = L \left(\left(\sum_{u \notin J} a_u u \right)^* \left(\sum_{u \notin J} a_u u \right) \right)$$

The right-hand side of this equation equals

$$a^* \begin{bmatrix} cI & B \\ B^* & A \end{bmatrix} a,$$

where a is the tuple of a_u , I is the identity matrix of suitable size (corresponding to the words u of degree $= d + 1$), and A and B are block matrices with

$$\begin{aligned}
 (28) \quad a^* \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} a &= L \left(\left(\sum_{u \in \langle x, x^* \rangle_d, u \notin J} a_u u \right)^* \left(\sum_{u \in \langle x, x^* \rangle_d, u \notin J} a_u u \right) \right) \\
 a^* \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} a &= L \left(\left(\sum_{u \in \langle x, x^* \rangle_d, u \notin J} a_u u \right)^* \left(\sum_{\deg(v)=d+1, v \notin J} a_v v \right) \right. \\
 &\quad \left. + \left(\sum_{\deg(v)=d+1, v \notin J} a_v v \right)^* \left(\sum_{u \in \langle x, x^* \rangle_d, u \notin J} a_u u \right) \right) \\
 &= 2 \operatorname{Re} L \left(\left(\sum_{u \in \langle x, x^* \rangle_d, u \notin J} a_u u \right)^* \left(\sum_{\deg(v)=d+1, v \notin J} a_v v \right) \right) \\
 a^* \begin{bmatrix} cI & 0 \\ 0 & 0 \end{bmatrix} a &= L \left(\left(\sum_{\deg(v)=d+1, v \notin J} a_v v \right)^* \left(\sum_{\deg(v)=d+1, v \notin J} a_v v \right) \right).
 \end{aligned}$$

The matrix A is positive definite by induction, so there exists c sufficiently large making the matrix

$$\begin{bmatrix} cI & B \\ B^* & A \end{bmatrix}$$

positive definite; e.g. any $c > 0$ with $\frac{1}{c}\|B\|^2 < \text{the minimal eigenvalue of } A$ will do. \blacksquare

6.1. *-Representations. Let V be a pre-Hilbert space, i.e., an \mathbb{R} -vector space with an inner product. A mapping π of $\mathbb{R}\langle x, x^* \rangle$ into the set of linear operators defined on V is said to be a **(unital) *-representation** of $\mathbb{R}\langle x, x^* \rangle$ on V if $\pi(1) = 1$ and it satisfies the familiar axioms:

$$\begin{aligned} \pi(\alpha_1 a_1 + \alpha_2 a_2)v &= \alpha_1 \pi(a_1)v + \alpha_2 \pi(a_2)v \\ \pi(a_1 a_2)v &= \pi(a_1)\pi(a_2)v \\ \langle \pi(a)v_1, v_2 \rangle &= \langle v_1, \pi(a^*)v_2 \rangle \end{aligned}$$

for every $a, a_1, a_2 \in \mathbb{R}\langle x, x^* \rangle$, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $v, v_1, v_2 \in V$. Let \mathcal{R} be the **class of all *-representations** of $\mathbb{R}\langle x, x^* \rangle$. Write $\partial_{\mathcal{R}}(\mathbb{R}\langle x, x^* \rangle)$ to be the set of pairs (π, v) such that π is a *-representation of $\mathbb{R}\langle x, x^* \rangle$ on some space V , and $0 \neq v \in V$. For every subset S of $\mathbb{R}\langle x, x^* \rangle$ write

$$V_{\mathcal{R}}(S) := \{(\pi, v) \mid \pi(s)v = 0 \text{ for every } s \in S\}.$$

For a subset T of $\partial_{\mathcal{R}}(\mathbb{R}\langle x, x^* \rangle)$, let

$$\mathcal{I}(T) := \{a \in \mathcal{A} \mid \pi(a)v = 0 \text{ for every } (\pi, v) \in T\}.$$

Note that $\mathcal{I}(T)$ is always a left ideal. Finally, for each subset S of $\mathbb{R}\langle x, x^* \rangle$ write

$$\sqrt[\mathcal{R}]{S} = \mathcal{I}(V_{\mathcal{R}}(S)).$$

Corollary 6.2. *Suppose the left monomial ideal $J \subseteq \mathbb{R}\langle x, x^* \rangle$ is real. Then $\sqrt[\mathcal{R}]{J} = J$.*

Proof. Let L be the linear functional constructed in Proposition 6.1. By the Gelfand-Naimark-Segal (GNS) construction, there exists $(\pi, v) \in \mathcal{R}$ such that $L(a) = \langle \pi(a)v, v \rangle$ for every $a \in \mathbb{R}\langle x, x^* \rangle$. (Recall that $V_{\pi} = \mathbb{R}\langle x, x^* \rangle / J$ considered as a vector space over \mathbb{R} with inner product $\langle p + J, q + J \rangle = L(q^*p)$, π is the left regular representation of $\mathbb{R}\langle x, x^* \rangle$ on V_{π} and $v = 1 + J$.) It follows that

$$J = \mathcal{I}(\{(\pi, v)\}). \quad \blacksquare$$

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